

Vacuum self similar anisotropic cosmologies in $F(R)$ –gravity

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Abstract. The implications from the existence of a proper Homothetic Vector Field (HVF) on the dynamics of vacuum anisotropic models in $F(R)$ gravitational theory are studied. The fact that *every* Spatially Homogeneous vacuum model is equivalent, formally, with a “flux”-free anisotropic fluid model in standard gravity and the induced power-law form of the functional $F(R)$ due to self-similarity enable us to close the system of equations. We found some new exact anisotropic solutions that arise as fixed points in the associated dynamical system. The non-existence of Kasner-like (Bianchi type I) solutions in proper $F(R)$ –gravity (i.e. $R \neq 0$) strengthens the belief that curvature corrections will prevent the shear influence into the past thus permitting an isotropic singularity. We also discuss certain issues regarding the lack of vacuum models of type III, IV, VII_h in comparison with the corresponding results in standard gravity.

1. Introduction

Geometric symmetries have been used widely enough within the context of General Relativity (GR) mainly because they lead to a significant reduction of the complexity of the Field Equations (FE). In addition, according to Noether’s observations, the existence of special geometric properties imply conservation laws and invariant quantities in the form of first integrals [1, 2]. The main disadvantages of these approaches is the lack of sound physical motivation by assuming a specific kind of geometric symmetry and the “information loss” from general models which leads to an incomplete view of the whole picture. A counterexample, so far, to this situation is the existence of a proper Homothetic Vector Field (HVF) admitted by the underlying geometry of a large set of cosmological or astrophysical models. Although the inspection of the role of homothetic (equivalently self-similar) models has not be exhausted, their importance is well established since transitively self-similar models represent the past and future attractors for the majority of evolving (non) vacuum models [3, 4, 5].

Motivated from the above facts it is of mathematical and physical interest to check if the above description of the asymptotic states of general models also holds for the so called modified theories of gravity. Considerable attention has been given to the extension of Einstein’s theory namely the quadratic theories of gravity in which curvature or/and Ricci scalar invariants contribute in the Lagrangian. From a physical point of view these theories appear to be an excellent enviroment to understand and solve various problems in contemporary cosmology like the accelerated phase of the Universe, the effect of quantum corrections to classical gravity or the asymptotic isotropisation near the initial singularity or at future times. The simplest version of the above alternative gravity theories is represented by the presence of a functional $F(R)$ in the action integral where R is the Ricci scalar. A vast number of studies has been appeared so far where the background geometry is described by the Robertson-Walker (RW) metric. It has been argued that $F(R)$ –theory could be a viable alternative of standard gravity solving many open questions of cosmological interest (see [6, 7, 8] for extensive reviews and bibliography).

Because $F(R)$ –theory is fourth order on the metric functions, the non-linearity of the FEs and the coupling between temporal and spatial dependence complexifies the analysis of the geometric and physical structure of the specific model. Therefore it seems natural that few results exist in the literature when we incorporate in $F(R)$ –gravity more general geometries than the RW. In cosmological scales the immediate departure from isotropy, while holding the spatial homogeneity, of the Friedmann-Lemaître-Robertson-Walker (FLRW) universe is represented by the Spatially Homogeneous (SH) or Bianchi geometries [9, 10, 11]. Then the FEs are

reduced to a coupled system of ordinary differential equations that must be satisfied by the anisotropy scale functions. The emergence of a power-law form of the metric (equivalently the existence of a proper HVF) is justified since it corresponds to fixed points of the associated dynamical system.

The details of the analysis described above can be found in the present paper as follows: in section 2 we review some basic results regarding the implications in the geometry and the dynamics of SH models from the existence of a proper HVF. An important ingredient of our discussion is the *symmetry inheritance* property of the timelike unit vector u^a which characterizes the SH geometries. The vacuum $F(R)$ -gravity within an anisotropic but homogeneous background is treated in Section 3. Due to the fact that every SH vacuum model in $F(R)$ -gravity is equivalent with a “flux”-free anisotropic fluid model in standard gravity where the effective “dissipative” tensor is expressed in terms of the shear of the comoving timelike congruence, the order of the resulting set of differential equations is reduced to three. In section 4 we specialize our study to vacuum self-similar models and derive the power-law form of the functional $F(R) \sim R^n$ w.r.t. the Ricci scalar. Accordingly we give the tetrad/expansion-normalized form of the effective FEs at the fixed points which are used to determine a certain number of equilibria. We conclude our analysis in section 5.

Throughout this paper we have used geometrized units such that $8\pi G = c = 1$ and the standard index conventions: spatial frame and coordinate indices are denoted by lower Greek letters $\alpha, \beta, \dots = 1, 2, 3$, lower Latin letters denote spacetime indices $a, b, \dots = 0, 1, 2, 3$.

2. Spatial homogeneity and homothetic symmetry

In cosmological setups the effects of the departure from the standard FLRW model, are studied by using its simplest generalization, the SH models. They are specified, in geometric terms, by requiring the existence of a G_3 Lie group of Killing Vector Fields (KVF) \mathbf{X}_α acting transitively on three-dimensional spacelike orbits \mathcal{S} . The conventional metric formalism [9, 10, 11] then is used in order to express the metric of the SH geometry in terms of the left-invariant 1-forms ω^α

$$ds^2 = -dt^2 + g_{\alpha\beta}(t)\omega_a^\alpha\omega_b^\beta dx^a dx^b \quad (2.1)$$

where $g_{\alpha\beta}(t)$ are smooth functions of the time coordinate and denote the spatial frame components of the induced three-dimensional metric, constant in each spacelike hypersurface $t = \text{const}$.

In addition, there exists a uniquely defined unit ($u^a u_a = -1$) timelike vector field u^a normal to the spatial foliations \mathcal{S} [12, 13]:

$$u_{[a;b]} = 0 = u_{a;b} u^b \Leftrightarrow \frac{1}{2} \mathcal{L}_u g_{ab} = u_{a;b} = \sigma_{ab} + \frac{\theta}{3} h_{ab} \quad (2.2)$$

where σ_{ab}, θ are the shear and expansion rates, associated with u^a , according to the standard 1+3 decomposition of an arbitrary timelike congruence and $h_{ab} = u_a u_b + g_{ab}$ is the projection tensor normal to u^a and represents the orthogonal metric of the instantaneous rest spaces of the timelike observers (with an obvious abuse of notation we can write $h_{ab} = g_{\alpha\beta}(t)\omega_a^\alpha\omega_b^\beta$). Because u^a is irrotational ($\omega^a = 0$) and geodesic ($\dot{u}^a \equiv u^a{}_{;b}u^b = 0$), there exists a time function $t(x^a)$ such that $u^a = \delta_t^a$ i.e. each value of t essentially represents the hypersurfaces \mathcal{S} .

On the other hand, one wishes to simplify further the geometry and the dynamics of the SH models by assuming the existence of a Homothetic Vector Field (HVF) \mathbf{H} which is defined as

$$\mathcal{L}_{\mathbf{H}}g_{ab} = 2\psi g_{ab} \quad (2.3)$$

where $\psi = \text{const.}$ essentially represents the (equal) time amplification and space dilation.

The major gain from such a simplification “assumption” is not only the reduction of the FEs to a system of algebraic equations but the fact that any solution of this system represent the asymptotic state of evolving SH (or even less symmetric) models. In the coordinates adapted to spatial homogeneity it can be shown [14, 15] that the HVF assumes the form $\mathbf{H} = \psi t \partial_t + H^\alpha(x^\beta) \partial_\alpha$.

On pure geometrical grounds, the existence of a HVF implies [1]

$$R_{abcd}H^d = F_{ab;c}, \quad \mathcal{L}_{\mathbf{H}}\Gamma_{cd}^a = 0, \quad \mathcal{L}_{\mathbf{H}}R_{bcd}^a = 0$$

$$\mathcal{L}_{\mathbf{H}}R_{ab} = 0, \quad \mathcal{L}_{\mathbf{H}}R = -2\psi R, \quad \mathcal{L}_{\mathbf{H}}G_{ab} = 0 \quad (2.4)$$

where $F_{ab} \equiv H_{[a;b]}$ is the homothetic bivector and $G_{ab} = R_{ab} - \frac{R}{2}g_{ab}$ is the Einstein tensor. Equations (2.3) and (2.4) mean that for a *proper* HVF (i.e. $\psi \neq 0$ and $\psi_{;a} = 0$) the geometrical quantities of the SH models scale $\sim t^p$. In particular the curvature scalar R satisfies

$$R = R_0 t^{-2}. \quad (2.5)$$

It should be also noticed that the invariance of the connection coefficients along the integral curves of a homothetic symmetry implies that the Lie and covariant derivatives commute [16]

$$\mathcal{L}_{\mathbf{H}}\nabla_a\Phi = \nabla_a\mathcal{L}_{\mathbf{H}}\Phi \quad (2.6)$$

for any scalar or tensorial quantity Φ .

Similar transformation mappings on the kinematical quantities of the unit timelike vector field u^a do not necessarily hold unless the homothetic symmetry *is inherited* by u^a i.e. $\mathcal{L}_{\mathbf{H}}u^a = -\psi u^a \Leftrightarrow \mathcal{L}_{\mathbf{H}}u_a = \psi u_a$. Although, in general, this is a consequence of the full FEs in the case of SH models it can be shown [14, 15] that the inheritance

property is an intrinsic feature of its geometric structure. As a result and using equation (2.6) we get

$$\mathcal{L}_{\mathbf{H}}u_{a;b} = \psi u_{a;b}, \quad \mathcal{L}_{\mathbf{H}}\sigma_{ab} = \psi \sigma_{ab}, \quad \mathcal{L}_{\mathbf{H}}\theta = -\psi \theta. \quad (2.7)$$

i.e. the kinematical quantities of the timelike congruence scale as t^{-1} (e.g. the expansion rate has the form $\theta = \theta_0 t^{-1}$).

The projection tensor also inherits the homothetic symmetry

$$\mathcal{L}_{\mathbf{H}}h_{ab} = 2\psi h_{ab} \quad (2.8)$$

which yields to the following useful identity:

$$\mathcal{L}_{\mathbf{H}}\left(h_a^c h_b^d - \frac{1}{3}h^{cd}h_{ab}\right) = 0 \quad (2.9)$$

where the $h_a^c h_b^d - \frac{1}{3}h^{cd}h_{ab}$ operator returns the u^a -normal and trace-free part of any second order symmetric tensor.

The above assumptions can be seen as purely geometrical therefore they are valid irrespective of the form of the FEs. One should expect that merging the dynamics of the system, embodied in the energy-momentum (EM) tensor T_{ab} , with the geometry of the SH models further restrictions in both sectors will appear. The standard way to analyze the dynamical content of any model is to decompose the associated EM tensor T_{ab} into irreducible parts w.r.t. u^a :

$$T_{ab} = \rho u_a u_b + p h_{ab} + q_{(a} u_{b)} + \pi_{ab} \quad (2.10)$$

where

$$\begin{aligned} \rho &\equiv T_{ab}u^a u^b & p &\equiv \frac{1}{3}T_{ab}h^{ab} \\ q_a &\equiv -h_a^c T_{cd}u^d & \pi_{ab} &\equiv \left(h_a^c h_b^d - \frac{1}{3}h^{cd}h_{ab}\right) T_{cd}. \end{aligned} \quad (2.11)$$

In any matter fluid model ρ , p , q_a and π_{ab} represent the energy density, the isotropic pressure, the heat flux vector field and the anisotropic pressure tensor respectively obeying the standard energy conditions [17]. Nevertheless the decomposition (2.10) remains true for every symmetric second order tensor and, as we shall see in the next section, will be used to set up the system of equations for the vacuum $F(R)$ -gravity in terms of dimensionless variables.

In order to visualize how the homothetic symmetry interacts with the dynamics, it is necessary to determine the effect of the former on the irreducible parts. Provided that $\mathcal{L}_{\mathbf{H}}u^a = -\psi u^a$ and $\mathcal{L}_{\mathbf{H}}T_{ab} = 0$ (which is the direct consequence of (2.4) and the FEs $G_{ab} = T_{ab}$), the quantities (2.11) are Lie transformed along H^a according to

$$\begin{aligned} \mathcal{L}_{\mathbf{H}}\rho &= -2\psi\rho, & \mathcal{L}_{\mathbf{H}}p &= -2\psi p \\ \mathcal{L}_{\mathbf{H}}q_a &= -2\psi q_a, & \mathcal{L}_{\mathbf{H}}\pi_{ab} &= 0 \end{aligned} \quad (2.12)$$

as we can verify using equations (2.7)-(2.9).

The first two equations imply that the equation of state of a self-similar fluid model is necessarily linear i.e. $p = w\rho$ where w is the (constant) state parameter. In the particular case of SH models $\rho = \rho_0 t^{-2}$ and $p = p_0 t^{-2}$.

3. Spatially homogeneous vacuum cosmologies in $F(R)$ - gravity

The effective action we are interested has the form

$$S = \int d^4x \sqrt{-g} [\mathcal{L}^{\text{mat}} + F(R)] \quad (3.1)$$

where $F(R)$ is an analytic function of the curvature scalar R .

Considering only the metric as the independent variable, the extrema of the action (3.1) give the effective FEs in the form [6, 7]

$$\Sigma_{ab} \equiv F_{,R}(R)R_{ab} - \frac{1}{2}F(R)g_{ab} - \nabla_a \nabla_b F_{,R}(R) + g_{ab} \nabla^2 F_{,R}(R) = T_{ab}^{\text{mat}} \quad (3.2)$$

where $F_{,R}(R) = \frac{dF}{dR}$ and T_{ab}^{mat} is the energy-momentum (EM) tensor representing the matter contributions in the dynamics of the model from a continuum system. It is easy to see that the tensor Σ_{ab} is divergence-free which implies the usual energy and momentum conservation. Taking the trace of (3.2)

$$\nabla^2 F_{,R} = \frac{1}{3} (2F - F_{,R}R + T^{\text{mat}}) \quad (3.3)$$

and substituting back to (3.2) we get

$$F_{,R}R_{ab} + \frac{1}{6}g_{ab} (F - 2F_{,R}R + 2T^{\text{mat}}) - \nabla_a \nabla_b F_{,R} = T_{ab}^{\text{mat}}. \quad (3.4)$$

The differential equation (3.3) imposes further constraints on the geometric structure of the model by restricting the functional form of $F(R)$. This will become more transparent when we shall include the self-similarity property.

We confine our study to SH vacuum models i.e. $T_{ab}^{\text{mat}} = 0$ and observe that, formally, the effective FEs (3.2) can be written in the familiar form of standard gravity as (provided that $F_{,R} \neq 0$)

$$G_{ab} = T_{ab}^{\text{eff}} \quad (3.5)$$

where

$$T_{ab}^{\text{eff}} \equiv -\frac{1}{6}g_{ab} \left(\frac{F}{F_{,R}} + R \right) + \frac{1}{F_{,R}} \nabla_a \nabla_b F_{,R}. \quad (3.6)$$

Because the curvature scalar is also SH, the tensor $\nabla_a \nabla_b F_{,R}$ can be written in terms of the kinematical quantities (2.2) of the timelike vector field u^a

$$\nabla_a \nabla_b F_{,R} = (F_{,R})^{\cdot\cdot} u_a u_b - (F_{,R})^{\cdot} \left(\sigma_{ab} + \frac{\theta}{3} h_{ab} \right) \quad (3.7)$$

where a dot “.” denotes differentiation w.r.t. u^a (equivalently w.r.t. the time coordinate).

Taking the trace of (3.7) and using (3.3) we get:

$$\nabla_a \nabla_b F_{,R} = \left[\frac{1}{3} (F_{,R} R - 2F) - (F_{,R})^\cdot \theta \right] u_a u_b - (F_{,R})^\cdot \left(\sigma_{ab} + \frac{\theta}{3} h_{ab} \right) \quad (3.8)$$

Equations (2.11) and (3.5) in conjunction with (3.8) imply that $F(R)$ vacuum models with an underlying SH geometry can be seen as *anisotropic* “fluid” models with “dynamical” quantities satisfying

$$\rho = \frac{1}{2} \left(R - \frac{F}{F_{,R}} \right) - \frac{(F_{,R})^\cdot}{F_{,R}} \theta \quad (3.9)$$

$$p = -\frac{1}{6} \left(R + \frac{F}{F_{,R}} \right) - \frac{1}{3} \frac{(F_{,R})^\cdot}{F_{,R}} \theta \quad (3.10)$$

$$q_a = -T_{cd}^{\text{eff}} u^c h_a^d = 0$$

$$\pi_{ab} = \left(h_a^c h_b^d - \frac{1}{3} h^{cd} h_{ab} \right) T_{cd}^{\text{eff}} = -\frac{(F_{,R})^\cdot}{F_{,R}} \sigma_{ab}. \quad (3.11)$$

As a result every SH vacuum model in $F(R)$ -gravity is equivalent with a “flux”-free anisotropic fluid model in standard gravity where the “dissipative” tensor satisfies the Eckart-Landau-Lifshitz relation [18].

In conventional fluid models of standard gravity there are no evolution equations for the isotropic pressure p and the anisotropic stress tensor π_{ab} . Therefore equations (3.3), (3.7), (3.8) and (3.9)-(3.11) *completely* determine the dynamics of the vacuum SH models in $F(R)$ -gravity provided that the functional form of $F(R)$ is known. One can then exploit the orthonormal frame formalism and the usage of expansion-normalized variables [19] to reformulate the FEs (3.5) as evolution equations of the shear and spatial curvature (described by the variables A_α and $N_{\alpha\beta}$ which identify each Bianchi type model). The evolution equations are subjected to algebraic constraints and can be used to study the intermediate and asymptotic behaviour of general SH models in $F(R)$ -gravity. We postpone this analysis for a future work. In the next section we use the set of equations of [19] as a guide in order to examine the existence of a particular self-similar model in $F(R)$ vacuum gravity and to give the corresponding exact solution whenever it exists.

4. The implications of self-similarity

The “flux”-free property and the presence of an additional contribution in the evolution equation of the shear as well as the *algebraic constraint* (3.9) are intrinsic

properties of the vacuum models in $F(R)$ - gravity that allow us to close the system of equations. On the other hand the identification of self similar models as equilibrium points in the dynamical phase space of general configurations shows that the existence or not of a proper HVF is crucial in understanding the structural form of the associated state space. Nevertheless we expect that the assumption of self-similarity will provide us with further, *algebraic* in nature, restrictions.

From the inheritance property of the shear (2.7), the invariance under Lie dragging of the anisotropic stress tensor (2.12) and equation (3.11) implies that

$$\mathcal{L}_{\mathbf{H}} \left[\frac{(F_{,R})}{F_{,R}} \right] = -\psi \frac{(F_{,R})}{F_{,R}}. \quad (4.1)$$

Expressing the last equation in coordinate form we can easily verify that the functional $F(R)$ satisfies

$$F(R) = F_0 R^n \quad (4.2)$$

where F_0 is an arbitrary constant and n is any real number.

Clearly, the power law behaviour of $F(R)$ is a *direct consequence of the scale invariant feature (self-similarity) of the SH vacuum models in modified gravity*. We note however that in evolving models the functional structure of $F(R)$ is more general and complicated. The established power law property of $F(R)$ permit us to determine the exact form of the “anisotropic” stress tensor π_{ab} . From equation (3.11) a straightforward calculation gives

$$\pi_{ab} = \frac{6(n-1)}{\theta_0} H \sigma_{ab} \equiv \tilde{A} H \sigma_{ab}. \quad (4.3)$$

We have employed the Hubble scalar defined as

$$H \equiv \frac{\theta}{3} = \frac{\theta_0}{3t}. \quad (4.4)$$

Although we have used the standard form for the FEs (3.5) we must emphasize that the definition of the effective “energy” density (3.9) or the effective isotropic “pressure” (3.10) reveal further constraints which one must take into account. In the particular case of self-similar models the induced linear dependence $p = w\rho$ implies that only one of them contains non trivial information and the other is satisfied identically. It follows from eqs (4.2)-(4.3) in conjunction with the trace $R = \rho - 3p = \rho(1 - 3w)$ of the effective FEs that the “energy” density satisfies the algebraic relation

$$\rho = \rho \frac{1 - 3w}{2} \frac{n - 1}{n} + 3H^2 \tilde{A}. \quad (4.5)$$

We observe that for $n = 1$ or equivalently $\tilde{A} = 0$ we reproduce the usual vacuum models in standard gravity ($\tilde{A} = 0 \Leftrightarrow \pi_{ab} = 0, \rho = p = 0$).

The key ingredient necessary to construct the associate dynamical system is to define the dimensionless time variable τ by:

$$\frac{dt}{d\tau} = \frac{1}{H}, \quad \frac{dH}{d\tau} = -(1+q)H \quad (4.6)$$

where q is the deceleration parameter and H is the Hubble scalar. Then expressing all the physical quantities in expansion-normalized form, the full set of equations at the fixed point becomes [19]

$$0 = -(2-q)\Sigma_{\alpha\beta} + 2\epsilon^{\mu\nu}{}_{(\alpha}\Sigma_{\beta)\mu}R_{\nu} - S_{\alpha\beta} + \tilde{A}\Sigma_{\alpha\beta} \quad (4.7)$$

$$0 = qN_{\alpha\beta} + 2\Sigma_{(\alpha}{}^{\mu}N_{\beta)\mu} + 2\epsilon^{\mu\nu}{}_{(\alpha}N_{\beta)\mu}R_{\nu} \quad (4.8)$$

$$0 = qA_{\alpha} - \Sigma_{\alpha}{}^{\mu}A_{\mu} + \epsilon_{\alpha}{}^{\mu\nu}A_{\mu}R_{\nu} \quad (4.9)$$

$$0 = (2q-1-3w)\Omega - \frac{1}{3}\tilde{A}\Sigma_{\mu\nu}\Sigma^{\mu\nu} \quad (4.10)$$

$$0 = 3\tilde{A}A^{\beta}\Sigma_{\alpha\beta} + \epsilon_{\alpha}{}^{\mu\nu}N_{\mu}{}^{\beta}\Sigma_{\beta\nu} \quad (4.11)$$

$$N_{\alpha}{}^{\beta}A_{\beta} = 0 \quad (4.12)$$

$$\Omega = 1 - \Sigma^2 - K \quad (4.13)$$

$$3A^{\beta}\Sigma_{\alpha\beta} - \epsilon_{\alpha}{}^{\mu\nu}N_{\mu}{}^{\beta}\Sigma_{\beta\nu} = 0 \quad (4.14)$$

$$\Omega = \Omega \frac{1-3wn-1}{2} \frac{n-1}{n} + \tilde{A}. \quad (4.15)$$

The above non linear system of algebraic equations is sufficient in order to locate any equilibria of the dynamical system in vacuum $F(R)$ -gravity. There are, however, two more restrictions which can be used as auxiliary relations due to their linearity. The first follows from (4.4) and (4.6) namely

$$\theta_0(1+q) = 3 \quad (4.16)$$

and the second is the definition of the dimensionless constant \tilde{A} which, in some sense, represents the deviation from standard gravity

$$\tilde{A} = \frac{6(n-1)}{\theta_0}. \quad (4.17)$$

The remaining freedom of a time-dependent spatial rotation permit us to choose the orthonormal tetrad to be the eigenframe of $N_{\alpha\beta}$ therefore the contracted form of Jacobi identities $N_{\alpha\beta}A^{\beta} = 0$ implies:

$$N_{\alpha\beta} = \begin{pmatrix} N_1 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{pmatrix}, \quad A_{\alpha} = A_1\delta_{\alpha}^1 \quad (4.18)$$

where the value of A_{α} distinguishes the models in class A ($A_{\alpha} = 0$) and class B ($A_1 \neq 0$).

In this case the angular velocity $R_\alpha = [R_1, R_2, R_3]$ of the spatial orthonormal frame will be specified from (4.7)-(4.17) as function of the shear variables. Furthermore the differential “versions” of (4.8) and (4.9) have a first integral and the component A_1 is expressed in the well known form:

$$A_1^2 = hN_2N_3. \quad (4.19)$$

Especially in type VI_h (where $N_2N_3 < 0$) the distinction between models with $h \neq -1/9$ and $h = -1/9$ is the reminiscence of the *exceptional* algebraic behaviour of the 0α -components of the FEs.

Throughout the rest of the present section we found some new *self similar SH vacuum* and *anisotropic* ($\sigma_{\alpha\beta} \neq 0$) models in *proper* $F(R)$ -gravity ($R \neq 0$) and give their local metric form using the results of [14, 15]. It should be noticed that the equilibria of the state space which correspond to vanishing shear (Robertson-Walker geometry) or the case where the curvature scalar is zero \ddagger (in our notation the constraint $R = 0$ is equivalent with a radiation-like fluid where the state parameter satisfies $w = 1/3$) are not of lesser importance to the qualitative study of SH vacuum models and will be reported in forthcoming works together with a detailed analysis of their asymptotical and intermediate behaviour.

SH models of Bianchi type II

Substituting $A_\alpha = 0$ and $N_2 = N_3 = 0$ back to (4.7)-(4.17) it follows that $R_\alpha = 0 = \Sigma_{\alpha\beta}$ ($\alpha \neq \beta$) i.e. the model is “diagonal” (the off-diagonal shear components vanishes). Solving the remaining equations we found the following fixed point

$$N_1 = \frac{6\sqrt{-(2n^2 - 2n - 1)(11n^2 - 20n + 8)}}{16n^2 - 25n + 10} \quad (4.20)$$

$$\Sigma_{22} = \Sigma_{33} = -\frac{2(2n^2 - 2n - 1)}{16n^2 - 25n + 10}. \quad (4.21)$$

$$q = 4\Sigma_{22} \quad (4.22)$$

The effective “energy” density and the state parameter are

$$\Omega = \frac{18(n-1)(17n^3 - 36n^2 + 24n - 4)}{(16n^2 - 25n + 10)^2} \quad (4.23)$$

$$w = -\frac{49n^3 - 84n^2 + 24n + 4}{3(17n^3 - 36n^2 + 24n - 4)}. \quad (4.24)$$

It is worth noticing that we do not require the positivity of Ω since the effective EM tensor does not represent an actual matter fluid. Nevertheless the “strong” energy

\ddagger We note that the solutions found in [20, 21] correspond either to a RW geometry or to a model with $R = 0$ (see also [22])

condition $\Omega > 0$ could be imposed in constructing a compact phase space for each model [23].

The above new exact anisotropic model of Bianchi type II in vacuum $F(R)$ -gravity admits the HVF

$$\mathbf{H} = \frac{3(n-2)}{2(2n^2-2n-1)}t\partial_t + x\partial_x + 2y\partial_y + z\partial_z \quad (4.25)$$

and the line element assumes the form

$$ds^2 = -dt^2 + \frac{1}{2}t^{2p_1}dx^2 + D(n)t^{2p_2}dy^2 - 2D(n)xt^{2p_2}dydz + \left[D(n)x^2t^{2p_2} - \frac{1}{2}t^{2p_1}\right]dz^2. \quad (4.26)$$

The p_1, p_2 indices and the polynom $D(n)$ satisfy the relations

$$p_1 = \frac{4n^2 - 7n + 4}{3(2-n)}, \quad p_2 = \frac{8n^2 - 11n + 2}{3(2-n)} \quad (4.27)$$

$$D(n) = \frac{22n^4 - 62n^3 + 45n^2 + 4n - 8}{9(n-2)^2}. \quad (4.28)$$

The negativity of the $\det(g)$ or equivalently the real values of the curvature variable N_1 imply

$$n \in \left(\frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{10}{11} - \frac{2\sqrt{3}}{11}\right) \vee \left(\frac{10}{11} + \frac{2\sqrt{3}}{11}, \frac{1}{2} + \frac{\sqrt{3}}{2}\right). \quad (4.29)$$

SH models of Bianchi type VI₀

Similarly with the previous type we get $R_\alpha = 0 = \Sigma_{\alpha\beta}$ ($\alpha \neq \beta$) and the curvature variables satisfy $N_{22} + N_{33} = 0$ therefore the fixed point corresponds to the subclass $N^\alpha_\alpha = 0$. The kinematical and dynamical variables of this model are given by

$$N_2 = -\frac{3\sqrt{-3(2n^2-2n-1)}|n-1|}{4n^2-7n+4} \quad (4.30)$$

$$\Sigma_{22} = \Sigma_{33} = \frac{2n^2-2n-1}{4n^2-7n+4} \quad (4.31)$$

$$\Omega = \frac{6(n-1)(5n^3-12n^2+9n-1)}{(4n^2-7n+4)^2} \quad (4.32)$$

$$w = -\frac{13n^3-24n^2+9n+1}{3(5n^3-12n^2+9n-1)}. \quad (4.33)$$

$$q = -2\Sigma_{22} \quad (4.34)$$

The generator of the self-similarity and the metric are

$$\mathbf{H} = \frac{n-2}{2n^2-2n-1}t\partial_t + \partial_x + 2y\partial_y \quad (4.35)$$

$$ds^2 = -dt^2 + t^2 D(n) dx^2 + t^{2p_1} e^{-2x} dy^2 + t^{2p_1} e^{2x} dz^2 \quad (4.36)$$

where

$$p_1 = \frac{2n^2 - 3n + 1}{2 - n} \quad (4.37)$$

$$D(n) = -\frac{(n-2)^2}{3(n-1)^2(2n^2 - 2n - 1)}. \quad (4.38)$$

This solution is well defined within the range

$$n \in \left(\frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{1}{2} + \frac{\sqrt{3}}{2} \right). \quad (4.39)$$

SH models of Bianchi type VI_h ($h \neq -1/9$)

In comparison with the standard gravity vacuum anisotropic cosmologies, the type VI_h models exhibit common features and significant differences as well. In particular we found that a solution exists only within the subclass satisfying $N^\alpha_\alpha = 0$, similar to the case $n = 1$, namely

$$N_2 = -\frac{3\sqrt{[h^2(n-2)^2 + 3(n-1)^2](-2n^2 + 2n + 1)}}{|3h^2(n-2) - 4n^2 + 7n - 4|} \quad (4.40)$$

$$\Sigma_{22} = \Sigma_{33} = -\frac{2n^2 - 2n - 1}{3h^2(n-2) - 4n^2 + 7n - 4}. \quad (4.41)$$

$$\Omega = \frac{6(h^2 + 1)(n-1)[3h^2n(n-2)^2 + 5n^3 - 12n^2 + 9n - 1]}{[3h^2(n-2) - 4n^2 + 7n - 4]^2} \quad (4.42)$$

$$w = -\frac{3h^2n(n-2)^2 + 13n^3 - 24n^2 + 9n + 1}{3[3h^2n(n-2)^2 + 5n^3 - 12n^2 + 9n - 1]}. \quad (4.43)$$

$$q = -2\Sigma_{22} \quad (4.44)$$

From (4.40) we deduce that the range of parameter n is the same like the type VI_0 . The FEs (3.2) and the usage of the associated self-similar metric [15] yield

$$\mathbf{H} = \frac{(2-n)(h^2+1)}{(2n^2-2n-1)(h-1)} t \partial_t + \frac{h+1}{1-h} y \partial_y + z \partial_z \quad (4.45)$$

$$ds^2 = -dt^2 + t^2 D(n) dx^2 + t^{2p_1} e^{-2x} dy^2 + t^{2p_2} e^{2\frac{h+1}{h-1}x} dz^2 \quad (4.46)$$

$$p_1 = \frac{h^2(n-2) - h(2n^2 - 2n - 1) - 2n^2 + 3n - 1}{(n-2)(h^2+1)} \quad (4.47)$$

$$p_2 = \frac{h^2(n-2) + h(2n^2 - 2n - 1) - 2n^2 + 3n - 1}{(n-2)(h^2+1)} \quad (4.48)$$

$$D(n) = -\frac{[(h^2 + 1)(n - 2)]^2}{(h - 1)^2 (2n^2 - 2n - 1) [h^2 (n^2 - 4n + 4) + 3(n - 1)^2]}. \quad (4.49)$$

In contrast with the corresponding model in standard gravity [15] the above solution does not admit a null gradient KVF therefore its Petrov type is D.

SH models of the exceptional Bianchi type VI_{-1/9}

Again the only non trivial solution satisfies $N^\alpha_\alpha = 0$ and the parameters of the model are

$$N_2 = -\frac{3\sqrt{6(-4n^2 + 6n - 1)}}{2|2n + 1|} \quad (4.50)$$

$$\Sigma_{22} = \Sigma_{33} = \frac{n - 2}{2n + 1} = \Sigma_{23} \quad (4.51)$$

$$\Sigma_{12} = \Sigma_{13} = -\frac{\sqrt{10(-8n^2 + 14n - 5)}}{2|2n + 1|} \quad (4.52)$$

$$R_1 = 0, \quad R_2 = \Sigma_{13} = -R_3 \quad (4.53)$$

$$\Omega = \frac{10(n - 1)(4n - 1)}{(2n + 1)^2} \quad (4.54)$$

$$w = \frac{1}{3(1 - 4n)}. \quad (4.55)$$

$$q = -2\Sigma_{22} \quad (4.56)$$

The exact form of this exceptional anisotropic model is

$$\mathbf{H} = \frac{1}{2 - n}t\partial_t + \frac{2}{5}y\partial_y + \frac{4}{5}z\partial_z \quad (4.57)$$

$$ds^2 = -dt^2 + t^2 D_1(n) dx^2 + t^{2p_1} e^{-2x} dy^2 + 2e^{x/2} t^{2p_1} dx dz + t^{2p_2} D_2(n) dz^2 \quad (4.58)$$

$$p_1 = \frac{2n + 1}{5}, \quad p_2 = \frac{4n - 3}{5} \quad (4.59)$$

$$D_1(n) = \frac{75}{32(n^2 - 4n + 4)} \quad (4.60)$$

$$D_2(n) = \frac{96(n^2 - 4n + 4)(4n^2 - 6n + 1)}{125(8n^2 - 14n + 5)}. \quad (4.61)$$

It can be verified that this solution does not admit also a covariantly constant null vector field which implies that the Petrov type is D (we recall that $N^\alpha_\alpha = 0 \Leftrightarrow \Sigma_{22} = \Sigma_{33}$).

Non existence of self similar SH models

Regarding the rest of the Bianchi types our study showed that there are no fixed points for the proper vacuum $F(R)$ -gravity. For example in type I the curvature variables are both zero ($A_\alpha = 0 = N_{\alpha\beta}$) and the shear evolution equation implies $R_\alpha = 0 = \Sigma_{\alpha\beta}$ ($\alpha \neq \beta$). It follows from the remaining set of equations that *there are no self-similar Bianchi type I anisotropic vacuum models* as we can easily verify using also the FEs (3.2) and the (diagonal) self similar three-dimensional metric components $g_{\alpha\beta} = \text{diag}(t^{p^1}, t^{p^2}, t^{p^2})$.

This conclusion signifies a direct deviation from the corresponding result of standard gravity where the Kasner circle plays a crucial role in the dynamics of anisotropic models (either vacuum or non vacuum) as past attractor [12]. In addition the non-existence of self-similar vacuum models of type III, IV and VII_h contradicts our expectations for a richer diversity of cosmological solutions due to the fourth order of the resulting FEs and one must take into account additional curvature invariants in the action integral [24, 25].

5. Discussion

In this paper we found a set of new exact power-law solutions which in principle can be used in order to understand the dynamics of anisotropic vacuum or fluid models in power-law $F(R)$ -gravity and analyze their asymptotic behaviour [18, 26, 27]. We emphasize that the analysis regarding the determination of the equilibria was not exhausted since we have excluded various special cases e.g. the shear-free models ($\sigma_{ab} = 0$), the vanishing of the Ricci scalar which represents a trivial solution of the FEs (3.2) or the stationary $q = -1$ cases. This does not mean that they are of minor importance for the qualitative study of SH vacuum models. For example it has been argued [22] that the flat RW model in $F(R)$ -gravity could be served as past attractor for anisotropic Bianchi type I models. Together with the non existence of proper Kasner-like solutions this conclusion imply that the curvature corrections dominate the shear influence and opens the possibility for an isotropic singularity which is not occurred in standard gravity.

Another worth noticing point is the non existence of type III, IV and VII_h self-similar models in $F(R)$ -gravity. A possible explanation for this “failure” is the fact that in standard gravity the vacuum solutions e.g. of type III and IV are of Petrov type N since they admit gradient null KVF's which essentially represent the repeated principal null direction of the Weyl tensor and is the characteristic feature of Kundt spacetimes [28]. A special subset are the Vanishing Scalar Invariant (VSI) spacetimes where the type III and IV solutions belong. Therefore we expect that including the self similarity property to models of $F(R)$ -gravity satisfying $R = 0$, algebraically special solutions could be found. On the other hand the existence of type VI_h

solutions for any non-zero value of the group parameter h which are of Petrov type D (i.e. they are not VSI as their counterparts in standard gravity) means that the effect of the curvature corrections depends on the underlying geometric structure of the model.

It will be interesting to extent the efforts for the determination of homothetic solutions in other categories of quadratic theories of gravity where the Lagrangian permits the inclusion of curvature invariants like $R_{ab}R^{ab}$ or/and $R_{abcd}R^{abcd}$. Among other important issues, that will show the level of departure from the well studied behaviour in standard gravity and the role that self-similar models play in modified theories. We believe that all these questions deserve further investigation and we intent to study them in subsequent works keeping in mind that a more sophisticated setup of the dynamical state space is required (see e.g. [29]).

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